Ejercicios

Ampliaciones 2019/20

I. Scimemi

Ejercicios de repaso

- **1-** Determinar todos los posibles valores de estos números complejos

$$2\,\mathrm{Log}\,(1-i);\;\; \mathrm{Log}\,(\sqrt{3}+i);\;\; \mathrm{Log}\,(4\,i);$$

$$2\ln(1-i) = \ln(1-i)^2 = \ln(\sqrt{2}e^{i(\frac{\pi}{4}+2n\pi)})^2 = \ln 2 + i\frac{\pi}{2} + in4\pi$$ En la primera hoja de Rieman n=0

- **2-** Sea C una circunferencia de radio 2 con centro en el origen. Calcular, la circunferencia en sentido anti-horario

$$\int_C \bar{z}\; dz$$

Poner $z = 2e^{i\theta} \rightarrow \bar{z} = 2e^{-i\theta}; dz = 2ie^{i\theta}d\theta ...$

- **3-**Utilizando el teorema de Cauchy calcular

$$\int_C dz\; \frac{e^{-z}}{z^2 - i\pi/2} \quad \text{con } C = \{|z| = 3\}$$

Los polos se encuentran en

In[3]:=
```
Simplify[Solve[z^2 - I π / 2 == 0, z]]
```

Out[3]= $\left\{\left\{z \to \left(-\frac{1}{2} - \frac{i}{2}\right)\sqrt{\pi}\right\}, \left\{z \to \left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{\pi}\right\}\right\}$

Los residuos valen

In[8]:=
```
Re1 = FullSimplify[Residue[e^(-z)/(z^2 - I π / 2), {z, (-1/2 - i/2) √π}]]
```

Out[8]= $-\frac{\left(\frac{1}{2} - \frac{i}{2}\right) e^{\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{\pi}}}{\sqrt{\pi}}$

In[9]:=
```
Re2 = FullSimplify[Residue[e^(-z)/(z^2 - I π / 2), {z, (1/2 + i/2) √π}]]
```

Out[9]= $\frac{\left(\frac{1}{2} - \frac{i}{2}\right) e^{\left(-\frac{1}{2}-\frac{i}{2}\right)\sqrt{\pi}}}{\sqrt{\pi}}$

Resultado

In[11]:=
```
FullSimplify[2 π I (Re1 + Re2)]
```

Out[11]= $(-1 - i)\, e^{\left(-\frac{1}{2}-\frac{i}{2}\right)\sqrt{\pi}} \left(-1 + e^{(1+i)\sqrt{\pi}}\right)\sqrt{\pi}$

- **4-Decir cual es el radio de convergencia de la serie**

$$f(z) = \sum_{k=2}^{+\infty} k(k+1)z^k.$$

y **calcular** f(i/2)

Observamos que

$$\sum_{k=1}^{+\infty} kz^{k-1} = \sum_{k=0}^{+\infty}(k+1)z^k = \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}$$

Entonces

$$\sum_{k=2}^{+\infty} k(k-1)z^{k-2} = \sum_{k=1}^{+\infty} k(k+1)z^{k-1} = \frac{d^2}{dz^2}\frac{1}{1-z} = \frac{2}{(1-z)^3}$$

$$f(z) = z\Big[\sum_{k=2}^{+\infty} k(k+1)z^{k-1}\Big] = z\Big[\sum_{k=1}^{+\infty} k(k+1)z^{k-1} - 2\Big] = z\Big[\frac{2}{(1-z)^3} - 2\Big].$$

$$f(\frac{i}{2}) = i\,[\Big(1-\frac{i}{2}\Big)^{-3} - 1] = \frac{6i-11}{2-11i} = -\frac{88+109i}{125}.$$

- **5-Calcular el desarrollo en serie de Taylor y/o Laurent de (algunas) de las siguientes funciones**

$i)\quad f(z)=\frac{1}{1-z}\,,\;\; z_0=3;$

$ii)\quad f(z)=\frac{1}{2-z^2}\,,\;\; z_0=0;$

$iii)\quad f(z)=\frac{1}{z}\,,\;\; z_0=1;$

$iv)\quad f(z)=\frac{1}{(z-1)(z-2)}\,,\;\; z_0=0;$

$v)\quad f(z)=\frac{2z+3}{z+1}\,,\;\; z_0=3;$

$vi)\quad f(z)=\frac{z}{(z^+1)(z-i)}\,,\;\; z_0=0;$

$vii)\quad f(z)=\frac{z^2}{(z-1)(z-2)}\,,\;\; z_0=0.$

i)

$$\frac{1}{1-z}=-\frac{1}{2}\frac{1}{1+(z-3)/2}=-\frac{1}{2}\sum_{k=0}^{+\infty}(-1)^k\left(\frac{z-3}{2}\right)^k=\sum_{k=0}^{+\infty}(-1)^{k+1}\frac{(z-3)^k}{2^{k+1}}.$$

ii)

$$\frac{1}{2-z^2}=\frac{1}{2[1-(z/\sqrt{2})^2]}=\sum_{k=0}^{+\infty}\frac{z^{2k}}{2^{k+1}}.$$

iii)

$$\frac{1}{z}=\frac{1}{1+(z-1)}=\sum_{k=0}^{+\infty}(-1)^k(z-1)^k.$$

iv)

$$\frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2} = \sum_{k=0}^{+\infty} z^k - \frac{1}{2}\sum_{k=0}^{+\infty}\frac{z^k}{2^k} = \sum_{k=0}^{+\infty}\left(1-\frac{1}{2^{k+1}}\right)z^k.$$

v)

$$2z+3 = 2(z-3)+9;$$

$$\frac{1}{z+1} = \frac{1}{4}\frac{1}{1+(z-3)/4} = \frac{1}{4}\sum_{k=0}^{+\infty}(-1)^k\left(\frac{z-3}{4}\right)^k = \sum_{k=0}^{+\infty}(-1)^k\frac{(z-3)^k}{4^{k+1}}$$

Entonces sigue que

$$\frac{2z+3}{z+1} = \frac{2(z-3)+9}{z+1} = 2\sum_{k=0}^{+\infty}(-1)^k\frac{(z-3)^{k+1}}{4^{k+1}} + 9\sum_{k=0}^{+\infty}(-1)^k\frac{(z-3)^k}{4^{k+1}} =$$

$$= \frac{9}{4} + \sum_{h=1}^{+\infty}\left[2\frac{(-1)^{h+1}}{4^h} + 9\frac{(-1)^h}{4^{h+1}}\right](z-3)^h = \frac{9}{4} + \sum_{h=1}^{+\infty}\frac{(-1)^h}{4^{h+1}}(z-3)^h.$$

vi)

$$f(z) = \frac{z}{(z+1)(z-i)} = \frac{1-i}{2}\frac{1}{z+1} + \frac{1+i}{2}\frac{1}{z-i}.$$

Si $|z| < 1$:

$$\frac{1}{z+1} = \sum_{k=0}^{\infty}(-z)^k;$$

$$\frac{1}{z-i} = \frac{i}{1+i\,z} = i\sum_{k=0}^{\infty}(-i\,z)^k = -\sum_{k=0}^{\infty}(-i)^{k+1}\,z^k;$$

Entonces por $|z| < 1$,

$$\frac{z}{(z+1)(z-i)} = \frac{1-i}{2}\sum_{k=0}^{\infty}(-1)^k z^k - \frac{1+i}{2}\sum_{k=0}^{\infty}(-i)^{k+1} z^k = \frac{1-i}{2}\sum_{k=1}^{\infty}((-1)^k - (-i)^k) z^k.$$

vii)

$$f(z) = \frac{z^2}{(z-1)(z-2)} = 1 - \frac{4}{2-z} + \frac{1}{1-z}$$

Se $|z| < 1$:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k;$$

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-z/2} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}.$$

Quindi

$$\frac{z^2}{(z-1)(z-2)} = 1 - 2\sum_{k=0}^{\infty}\frac{z^k}{2^k} + \sum_{k=0}^{\infty} z^k = 1 + \sum_{k=0}^{\infty}\left(1 - \frac{1}{2^{k-1}}\right) z^k = \sum_{k=1}^{\infty}\left(1 - \frac{1}{2^{k-1}}\right) z^k$$

• 6-Calcular el desarrollo en serie Taylor y/o Laurent de

$$f(z) = \frac{1}{(z-2)^2}$$

in $\{z \in \mathbb{C} : |z| < 2\}$ e in $\{z \in \mathbb{C} : |z| > 2\}$.

Recordamos que $|z| < 2$:

$$\frac{1}{2-z} = \frac{1}{2}\frac{1}{1-z/2} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}.$$

Y derivando termino a termino

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{d}{dz}\frac{z^k}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty}k\frac{z^{k-1}}{2^k} = \sum_{k=0}^{\infty}(k+1)\frac{z^k}{2^{k+2}}$$

Entonces

$$\frac{1}{(z-2)^2} = \frac{d}{dz}\frac{1}{2-z} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{d}{dz}\frac{z^k}{2^k} = \sum_{k=0}^{\infty}(k+1)\frac{z^k}{2^{k+2}}.$$

Recordamos que $|z| > 2$:

$$\frac{1}{2-z} = -\frac{1}{z}\frac{1}{1-2/z} = -\frac{1}{z}\sum_{k=0}^{\infty}\frac{2^k}{z^k} = -\sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}}.$$

$$-\sum_{k=0}^{\infty}\frac{d}{dz}\frac{2^k}{z^{k+1}} = \sum_{k=0}^{\infty}(k+1)\frac{2^k}{z^{k+2}}$$

$$\frac{1}{(z-2)^2} = \frac{d}{dz}\frac{1}{2-z} = \sum_{k=0}^{\infty}(k+1)\frac{2^k}{z^{k+2}}.$$

• 7-Determinar el modulo de la función **f** y para que valores de z la función vale 1

$$f(z) = \exp(\exp(z)).$$

Entonces

$$f(z) = \exp(\exp(z)) = e^{e^x(\cos(y)+i\sin(y))} = e^{e^x\cos(y)}e^{ie^x\sin(y)}.$$

Es decir

$$|f(z)| = e^{e^x\cos(y)}.$$

Entonces

$$e^{e^x\cos(y)} = 1 \iff e^x\cos(y) = 0 \iff y = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}; \forall x \in \mathbb{R}.$$

Es decir

$$|f(z)| = 1 \iff z = x + i\left(\frac{\pi}{2} + k\pi\right),\ \forall x \in \mathbb{R},\ k \in \mathbb{Z}.$$

- 8-**Escribir la parte principal en el desarrollo de Laurent/Taylor de las funciones**

$$\frac{\sin(z^4)}{z}, \quad \frac{1-\exp(-z)}{z}, \quad \frac{\exp(-1/z^2)}{z}, \quad z^3 \sin\left(\frac{1}{z}\right)$$

1. $$\frac{\sin(z^4)}{z} = \frac{1}{z}\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} z^{4(2k+1)} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} z^{8k+3}.$$ Y la singularidad en z=0 es eliminable

2. $$\frac{1-\exp(-z)}{z} = \frac{1}{z}\left(1-\sum_{k=0}^{+\infty} \frac{(-1)^k z^k}{k!}\right) = -\frac{1}{z}\sum_{k=1}^{+\infty} \frac{(-1)^k z^k}{k!} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+1)!} z^k.$$ Y la singularidad en z=0 es eliminable

3. $$\frac{\exp(-1/z^2)}{z} = \frac{1}{z}\sum_{k=0}^{+\infty} \frac{(-1)^k}{k! z^{2k}} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \frac{1}{z^{2k+1}}.$$ Y la singularidad en z=0 es Esencial

4. $$z^3 \sin\left(\frac{1}{z}\right) = z^3 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)! z^{2k+1}} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)! z^{2k-2}}.$$ Y la singularidad en z=0 es Esencial

- 9-**Utilizando el método de los residuos calcular las integrales**

$$a)\quad \int_{+\gamma} \frac{\mathrm{d}z}{z^2-1}\,, \qquad \text{dove} \quad \gamma:\ |z|=2;$$

$$b)\quad \int_{+\gamma} \frac{z^2}{(z^2+1)(z-2)}\mathrm{d}z\,, \qquad \text{dove} \quad \gamma:\ |z|=3;$$

$$c)\quad \int_{+\gamma} \frac{\sin(z+1)}{z(z+1)}\mathrm{d}z\,, \qquad \text{dove} \quad \gamma:\ |z|=3;$$

$$d)\quad \int_{+\gamma} \frac{z(z+1)}{\sin(z+1)}\mathrm{d}z\,, \qquad \text{dove} \quad \gamma:\ |z|=3;$$

a)

$$\int_{+\partial D} f(z)dz = 2\pi i[\mathrm{Res}(f,1)+\mathrm{Res}(f,-1)] = 2\pi i[\frac{1}{2}-\frac{1}{2}] = 0.$$

b)

$$\int_{+\partial D} f(z)dz = 2\pi i[\mathrm{Res}(f,i)+\mathrm{Res}(f,-i)+\mathrm{Res}(f,2)]$$

$$= 2\pi i[\frac{i}{2(i-2)}+\frac{i}{2(i+2)}+\frac{4}{5}] = 2\pi i.$$

c)

$$\int_{+\partial D} f(z)dz = 2\pi i[\mathrm{Res}(f,0)+\mathrm{Res}(f,-1)] = 2\pi i\mathrm{Res}(f,0) = 2\pi i\sin(1).$$

d)

$$\int_{+\partial D} f(z)dz = 2\pi i[\mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_0)]$$

$$= 2\pi i\mathrm{Res}(f,z_1) = 2\pi i[\pi(1-\pi)] = 2\pi^2[1-\pi]i.$$

- 10-**Estudiar que tipo de singularidad en el punto z=infinito para las funciones**

$a)\quad \dfrac{z}{z^2+1},$

$b)\quad 1-\dfrac{1}{z}+\dfrac{1}{z^3},$

$c)\quad z^2 e^{1/z},$

$d)\quad \dfrac{1}{z^2+1},$

$e)\quad \dfrac{z^6}{(z^2+1)(z^2-4)},$

$f)\quad \dfrac{1}{\sin 1/z}.$

- 11-**Calcular las siguientes integrales (los circuitos están orientados siempre en sentido antihorario)**

$$\int_{+\gamma} \frac{3z+1}{z(z-1)^3}\, dz, \qquad \text{con } \gamma : |z| = 2$$

$$\int_{+\gamma} \frac{z^3}{2\, z^4+1}\, dz, \qquad \text{con } \gamma : |z| = 2$$

$$\int_{+\gamma} \frac{e^{1/(z-1)}}{z-2}\, dz, \qquad \text{con } \gamma : |z| = 4$$

$$\int_0^{+\infty} \frac{\cos(ax)-\cos(bx)}{x^2} dx\,; \quad a, b \geq 0.$$

Quanto cum dolore

Clases de ejercicios

Ampliaciones

Calcular la integral

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^4+81} dx\,.$$

siendo $\omega > 0$.

Numerador

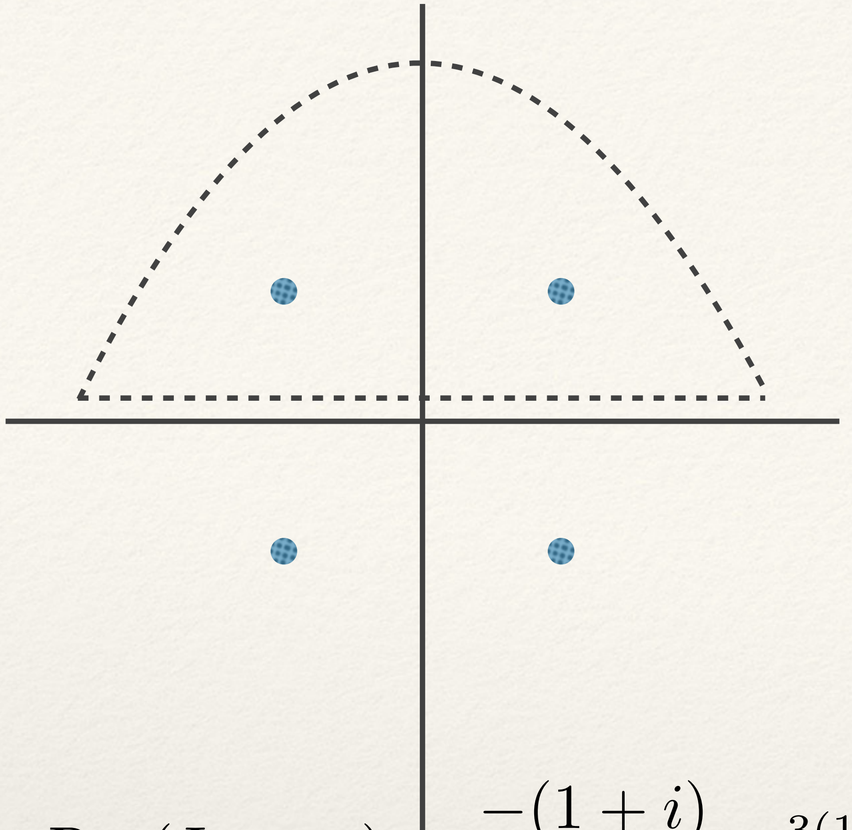
$$\cos(\omega x) = \frac{1}{2}(e^{i\omega x} + e^{-i\omega x})$$

Denominador

$$x^4+81=0 \rightarrow x^4 = -3^4 = 3^4 e^{i\pi + 2k\pi} \rightarrow x = 3e^{i\pi/4 + k\pi/2} \text{ con } \quad k = 0, 1, -1, -2$$

$$x_1 = \frac{3}{\sqrt{2}}(1+i); \quad x_2 = \frac{3}{\sqrt{2}}(-1+i); \quad x_3 = \frac{3}{\sqrt{2}}(1-i); \quad x_4 = \frac{3}{\sqrt{2}}(-1-i);$$

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^4+81} = \frac{1}{2}\Big[f(\omega) + f(-\omega)\Big] \text{ con } f(\omega) = \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^4+81}$$



$$f(\omega) = 2\pi i\left[\text{Res}(Int, x_1) + \text{Res}(Int, x_2)\right]$$

$$\text{Res}(Int, x_1) = \frac{-(1+i)}{108\sqrt{2}} e^{-3(1-i)\omega/\sqrt{2}} = \frac{-(1+i)}{108\sqrt{2}} e^{-3\omega/\sqrt{2}}(\cos\frac{3\omega}{\sqrt{2}} + i\sin\frac{3\omega}{\sqrt{2}})$$

$$\text{Res}(Int, x_2) = \frac{(1-i)}{108\sqrt{2}} e^{-3(1+i)\omega/\sqrt{2}} = \frac{(1-i)}{108\sqrt{2}} e^{-3\omega/\sqrt{2}}(\cos\frac{3\omega}{\sqrt{2}} - i\sin\frac{3\omega}{\sqrt{2}})$$

$$f(\omega) = \frac{(\pi)}{27\sqrt{2}} e^{-3\omega/\sqrt{2}}(\cos\frac{3\omega}{\sqrt{2}} + \sin\frac{3\omega}{\sqrt{2}})$$ [redacted]

$$\frac{1}{2}\Big[f(\omega) + f(-\omega)\Big] = \frac{(\pi)}{27\sqrt{2}}\left[\cos\frac{3\omega}{\sqrt{2}}\cosh\frac{3\omega}{\sqrt{2}} + \sin\frac{3\omega}{\sqrt{2}}\sinh\frac{3\omega}{\sqrt{2}}\right]$$

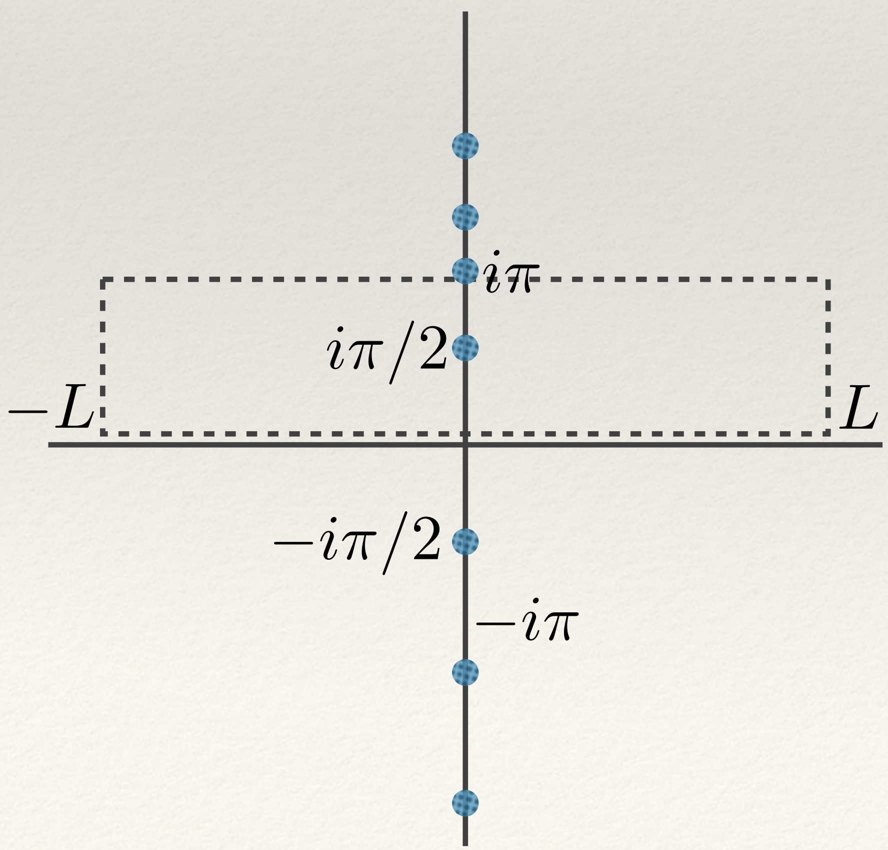
III Sea C el circuito de forma rectangular con vertices L, $L+i\pi$,$-L+i\pi$, $-L$ (siendo L>0). Calcular la integral

$$PV\int_C \frac{z}{\sinh 2z}dz \ .$$

¿Hay algun cambio en el resultado en el límite $L \to \infty$?

Ceros del denominador $\quad z = \frac{i\pi}{2}k$ con $k = 0, \pm 1, \pm 2, \ldots$

El integrando no tiene polo en $z = 0$ porqué $\lim_{z\to 0} \frac{z}{\sinh 2z} = \frac{1}{2}$



$$\text{Res}\left(\frac{z}{\sinh 2z}, i\frac{\pi}{2}\right) = -i\frac{\pi}{4} \qquad \text{Res}\left(\frac{z}{\sinh 2z}, i\pi\right) =$$

$$\int_C \left(\frac{z}{\sinh 2z}\right) = 2\pi i \text{Res}\left(\frac{z}{\sinh 2z}, i\frac{\pi}{2}\right) + 2\pi i \text{Res}\left(\frac{z}{\sinh 2z}, i\pi\right)$$

$$\int_0^\infty \frac{\cos 2x}{x^2+3}\, dx$$

$$\int_0^\infty \frac{\cos 2x}{x^2+3}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos 2x}{x^2+3}\, dx = \frac{1}{2}\mathrm{Re}\int_{C_R} \frac{e^{2iz}}{z^2+3}\, dz$$

$$\frac{1}{2}\int_{C_R} \frac{e^{2iz}}{z^2+3}\, dz = \frac{1}{2} 2\pi i \mathrm{Res}_{z=i\sqrt{3}}\left[\frac{e^{2iz}}{(z+i\sqrt{3})(z-i\sqrt{3})}\right] = \pi i \frac{e^{(-2\sqrt{3})}}{2i\sqrt{3}} = \pi \frac{e^{-(2\sqrt{3})}}{2\sqrt{3}}$$

$$\int_{|z|=2} e^{\frac{2}{z}} \left(4z^2 + 3z^3\right)\, dz$$

$$\int_{|z|=2} e^{\frac{2}{z}} \left(4z^2 + 3z^3\right)\, dz = \int_{|z|=2} \left(4z^2 + 3z^3\right) \left(1 + \frac{2}{z} + \frac{2^2}{2!z^2} + \frac{2^3}{3!z^3} + \frac{2^4}{4!z^4} + \ldots\right)\, dz$$

$$= \int_{|z|=2} 4z^2 \frac{2^3}{3!z^3} + 3z^3 \frac{2^4}{4!z^4} dz = 2\pi i \left(4\frac{2^3}{3!} + 3\frac{2^4}{4!}\right) = i\pi \frac{44}{3}$$

Hallar el recinto de convergencia de las siguientes series

$$\sum_{n=1}\left(z^n + \frac{1}{3^n z^n}\right)$$

$$\sum_n z^n \text{ es una serie geometrica que converge por } |z| < 1$$

$$\sum_n (3z)^{-n} \text{ es una serie geometrica que converge por } |3z|^{-1} < 1 \iff |z| > \frac{1}{3}$$

$$\sum_{n=1}\left(z^n + \frac{1}{3^n z^n}\right) \text{ converge por } \frac{1}{3} < |z| < 1$$

Hallar el recinto de convergencia de las siguientes series

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{1 - z^n} \right) .$$

- Si $|z| < 1$ ..y "n" suficientemente grande $\dfrac{z^n}{1 - z^n} \sim z^n$ .. y la serie converge

Si $|z| > 1$ ..y "n" suficientemente grande $\dfrac{z^n}{1 - z^n} \sim -1$ .. y la serie diverge

Calcular la parte principal de las siguientes funciones alrededor de los puntos indicados

$$z^2 e^{1/z}; \quad a) z = 0 \quad b) z = \infty$$

$$\frac{e^z - 1}{z^2 \ln(1-z)}; \quad a) z = 0 \quad b) z = -\infty$$

z=0

$$z^2 e^{1/z} = z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \dots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \dots$$

$$= z^2 + z + \frac{1}{2} + \sum_{n=1} \frac{1}{(n+2)!z^n}$$

$$\frac{e^z - 1}{z^2 \ln(1-z)} = \frac{1}{z^2}\left(z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \dots\right) \frac{1}{\left[(-)(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \dots)\right]}$$

$$= \frac{-1}{z^2}\left(1 + \frac{1}{2!}z + \frac{1}{3!}z^2 + \frac{1}{4!}z^3 + \dots\right) \frac{1}{\left[(1 + \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \dots)\right]}$$

$$= \frac{-1}{z^2}(1 + \mathcal{O}(z^2)) \Longrightarrow \text{Parte principal} = -\frac{1}{z^2}$$

Calcular la parte principal de las siguientes funciones alrededor de los puntos indicados

$$z^2 e^{1/z}; \quad a) z = 0 \quad b) z = \infty$$
$$\frac{e^z - 1}{z^2 \ln(1-z)}; \quad a) z = 0 \quad b) z = -\infty$$

$z = \infty$ Cambiamos variable $z \to z' = 1/z$

..y desarrollamos alrededor de $z' = 0$

$$z^2 e^{1/z} = \frac{1}{z'^2} e^{z'} = \frac{1}{z'^2} + \frac{1}{z'} + \frac{1}{2} + \mathcal{O}(z'^1) = z^2 + z + \frac{1}{2} + \mathcal{O}(z^{-1})$$

$z = -\infty$ Cambiamos variable $z \to z' = 1/z$

..y desarrollamos alrededor de $z' = 0$

$$\frac{e^z - 1}{z^2 \ln(1-z)} = \frac{z'^2 (e^{-1/z'} - 1)}{\ln(z' - 1) - \ln z'} = \frac{z'^2 (e^{-1/z'} - 1)}{i\pi + \mathcal{O}(z') - \ln z'}$$

..en este caso tenemos un corte

$$PV\int_{-\infty}^{\infty}\frac{x\cos x}{x^2+7x+10}$$

Polos simples en x=-2 y x=-5

Podemos calcular

$$PV\int_{-\infty}^{\infty}\frac{x\cos\omega x}{x^2+7x+10}\bigg|_{\omega=1} = \frac{1}{2}(f(\omega)+f(-\omega))\bigg|_{\omega=1} = \mathrm{Re} f(\omega)|_{\omega=1}$$

$$f(\omega) = PV\int_{-\infty}^{\infty}\frac{xe^{i\omega x}}{x^2+7x+10}$$

$$f(\omega) = \pi i\,[\mathrm{Res}(Int,-2)+\mathrm{Res}(Int,-5)] = \pi i\left[\frac{-2}{3}e^{-2i\omega}+\frac{5}{3}e^{-5i\omega}\right]$$

$$= \pi i\left[\frac{-2}{3}(\cos(2\omega)-i\sin(2\omega))+\frac{5}{3}(\cos(5\omega)-i\sin(5\omega))\right]$$

$$\mathrm{Re} f(\omega) = \pi\left[\frac{-2}{3}\sin(2\omega)+\frac{5}{3}\sin(5\omega)\right]$$

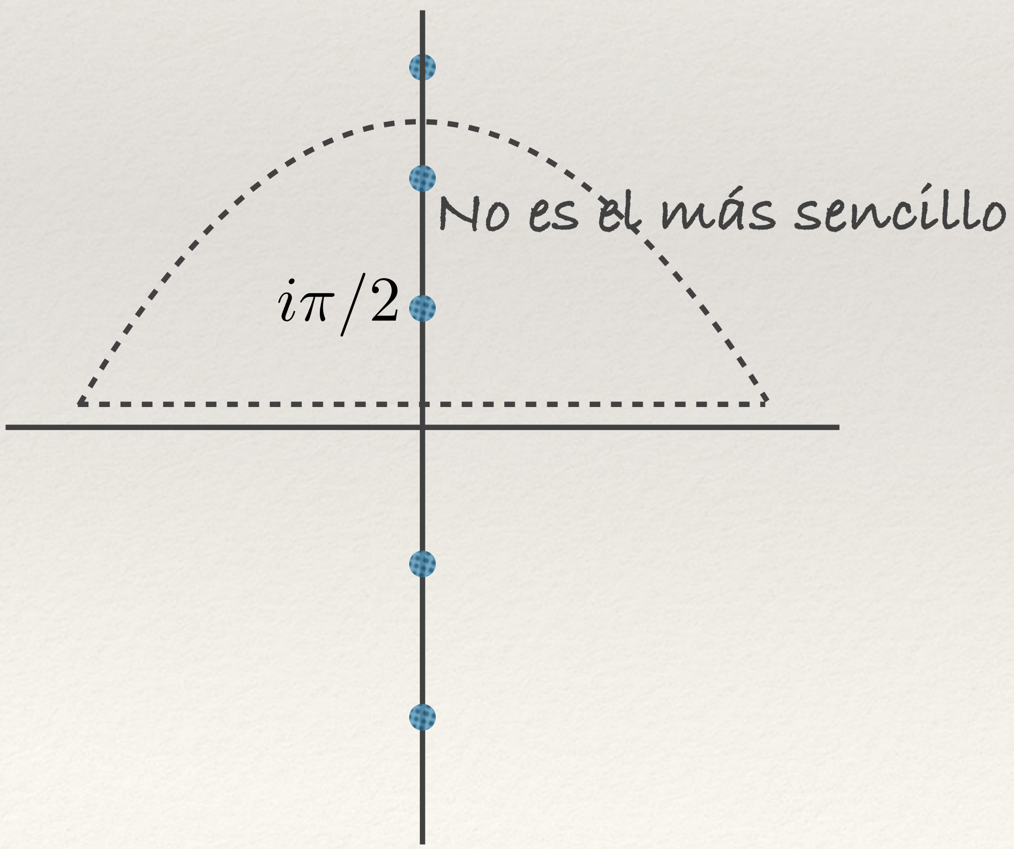
$$PV\int_{-\infty}^{\infty}\frac{x\cos\omega x}{x^2+7x+10}\bigg|_{\omega=1} = \pi\left[\frac{-2}{3}\sin(2)+\frac{5}{3}\sin(5)\right]$$

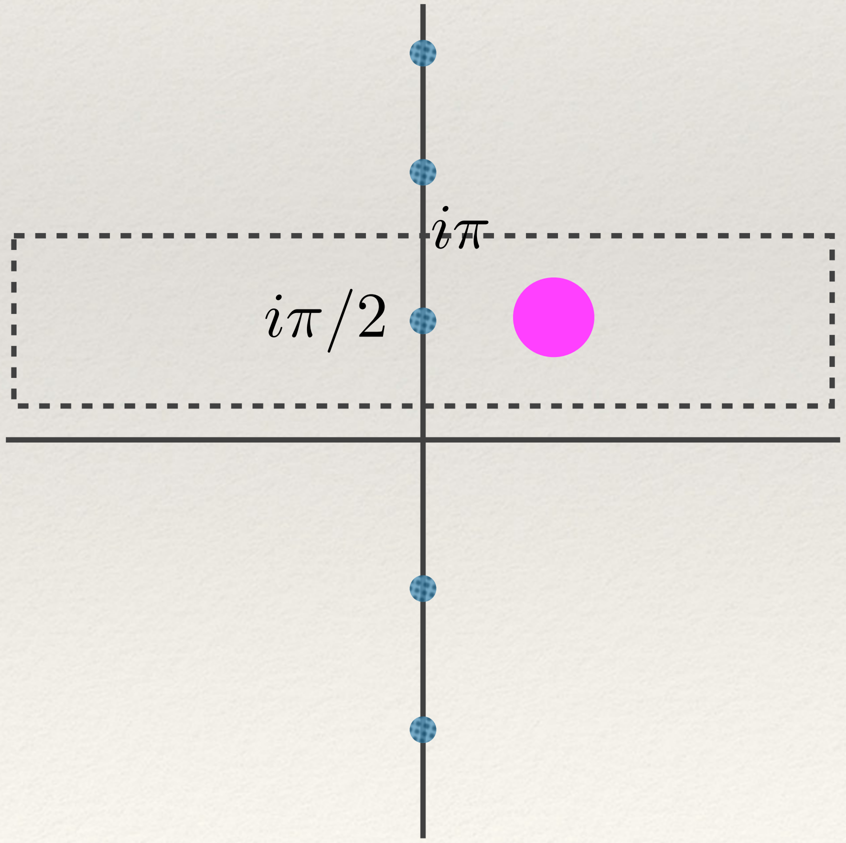
$$\int_0^\infty \frac{\cos \omega t}{\cosh t} dt; \ \ \omega > 0.$$

$$\int_0^\infty \frac{\cos \omega t}{\cosh t} = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos \omega t}{\cosh t}$$

$$\int_{-\infty}^\infty \frac{\cos \omega t}{\cosh t} = \mathrm{Re} \int_{-\infty}^\infty \frac{e^{i\omega t}}{\cosh t}$$

Polos simples en $i\frac{\pi}{2} \pm k i \pi, \quad k = 0, 1, 2, 3..$





La integral sobre el circuito rectangular da

$$2\int_{-\infty}^{\infty} dt\, \frac{e^{i\omega t}}{\cosh t} = 2\pi i\, \mathrm{Res}(Int, i\pi/2) = 2\pi i(-i)e^{-\pi\omega/2}$$

$$\int_{-\infty}^{\infty} dt\, \frac{e^{i\omega t}}{\cosh t} = \pi e^{-\pi\omega/2}$$

$$\int_{-\infty}^{\infty} dt\, \frac{\cos \omega t}{\cosh t} = \pi e^{-\pi\omega/2}$$

Clase II

- 10-**Estudiar que tipo de singularidad en el punto z=infinito para las funciones**

$a)\quad \dfrac{z}{z^2+1},$

$b)\quad 1-\dfrac{1}{z}+\dfrac{1}{z^3},$

$c)\quad z^2 e^{1/z},$

$d)\quad \dfrac{1}{z^2+1},$

$e)\quad \dfrac{z^6}{(z^2+1)(z^2-4)},$

$f)\quad \dfrac{1}{\sin 1/z}.$

Para estudiar la singularidad en z=infinito hay escribir estas funciones en la variable z′=1/z y desarrollar alrededor de z′=0

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z/(z^2+1) /. z -> 1/z
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$$\frac{1}{\left(1+\frac{1}{z^2}\right)z}$$

```
Series[1/((1+1/z^2) z), {z, 0, 2}]
```

$$z + O[z]^3$$

No hay singularidades

$$1 - \frac{1}{z} + \frac{1}{z^3} \to 1 + z' + z'^3, \text{ no hay singularidades en z'=0}$$

$$z^2 e^{1/z} \to z'^{-2} e^{z'} = z'^{-2} + z'^{-1} + \sum_{n=0} \frac{z'^n}{(n+2)!}$$

$$= z^2 + z^1 + \sum_{n=0} \frac{1}{z^n (n+2)!}$$

$$\frac{1}{\sin(1/z)} = \frac{1}{\sin z'}$$ Recordamos ahora que $$\lim_{z' \to 0} \frac{z'}{\sin z'} = 1 \to \frac{1}{\sin z'} = \frac{1}{z'} + \mathcal{O}(z'^0)$$

$$\frac{1}{\sin(1/z)} \to z + \mathcal{O}(z'^0)$$

Y hay singularidades

- 11-**Calcular las siguientes integrales (los circuitos están orientados siempre en sentido antihorario)**

$$\int_{+\gamma} \frac{3z+1}{z(z-1)^3}\,dz, \qquad \text{con } \gamma : |z| = 2$$

$$\int_{+\gamma} \frac{z^3}{2\,z^4+1}\,dz, \qquad \text{con } \gamma : |z| = 2$$

$$\int_{+\gamma} \frac{e^{1/(z-1)}}{z-2}\,dz, \qquad \text{con } \gamma : |z| = 4$$

$$\int_0^{+\infty} \frac{\cos(ax)-\cos(bx)}{x^2}dx\,; \quad a, b \geq 0.$$

$$\int_{+\gamma} \frac{3z+1}{z(z-1)^3}\, dz,$$ con $\gamma : |z| = 2$

Tenemos un polo simple en z=0 y un polo triple en z=1. El resultado es

$$\text{Res}(\frac{3z+1}{z(z-1)^3}, 0) = \lim_{z\to 0} \frac{3z+1}{z(z-1)^3}\, z = -1.$$

$$\text{Res}(\frac{3z+1}{z(z-1)^3}, 1) = \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2}(\frac{3z+1}{z(z-1)^3}\,(z-1)^3) = 1.$$

$$\int_{+\partial D} \frac{3z+1}{z(z-1)^3}\, dz = 2\pi\, i(\text{Res}(f,0) + \text{Res}(f,1)) = 0.$$

$$\int_{+\gamma} \frac{z^3}{2\,z^4+1}\,dz, \qquad \text{con } \gamma : |z| = 2$$

$$z^4 = -\frac{1}{2} = \frac{1}{2}\mathrm{e}^{(1+2k)\pi i} \to z_k = \frac{1}{\sqrt[4]{2}}\mathrm{e}^{\frac{(1+2k)}{4}\pi i}, \quad k = 0, 1, 2, 3.$$

$$\mathrm{Res}(\frac{z^3}{2\,z^4+1}, z_k) = \lim_{z \to z_k} \frac{z^3\,(z-z_k)}{2\,z^4+1} = \text{de l'Hopital} = \lim_{z \to z_k}\left(\frac{1}{2} - \frac{3z_k}{8z}\right) = \frac{1}{8}.$$

$$\int_{+\partial D} \frac{z^3}{2\,z^4+1}\,dz = 2\pi\,i \sum_{k=0}^{3} \mathrm{Res}(f, z_k) = 2\pi\,i\,4\,\frac{1}{8} = \pi\,i.$$

$$\int_{+\gamma} \frac{e^{1/(z-1)}}{z-2}\, dz, \qquad \text{con } \gamma : |z| = 4$$

$$\int_{+\partial D} \frac{e^{1/(z-1)}}{z-2}\, dz = 2\pi i(\operatorname{Res}(f,1) + \operatorname{Res}(f,2)).$$

$$\operatorname{Res}(f,2) = e.$$

$$\operatorname{Res} \frac{e^{1/(z-1)}}{z-2}\Big|_{z=1} \overset{(\text{ poniendo } z'=z-1)}{=} \operatorname{Res} \frac{-e^{1/z'}}{1-z'}\Big|_{z'=0}$$

$$= -\operatorname{Res} \sum_{n=0} \frac{1}{z'^n n!} \sum_{m=0} z'^m \Big|_{z'=0}$$

El residuo es el coeficiente del polo simple: n=m+1

$$= -\operatorname{Res} \sum_{n=0} \sum_{m=0} \frac{z'^{m-n}}{n!} \Big|_{z'=0}$$

$$= -\operatorname{Res} \sum_{m=0} \frac{z'^{-1}}{(m+1)!} \Big|_{z'=0} = -\sum_{m=0} \frac{1}{(m+1)!}$$

$$= -\left(\sum_{m=0} \frac{1}{(m)!} - 1 \right) = -(e-1)$$

Sumando todo el resultado final es $\displaystyle\int_{+\gamma} \frac{e^{1/(z-1)}}{z-2}\, dz = 2\pi i.$

$$\int_0^{+\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx\,; \quad a, b \geq 0.$$

$$\int_0^{\infty} dx\, \frac{\cos(ax) - \cos(bx)}{x^2} = \frac{1}{2} \int_{-\infty}^{\infty} dx\, \frac{\cos(ax) - \cos(bx)}{x^2}$$
$$= \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} dx\, \frac{e^{iax} - e^{ibx}}{x^2}$$

$$\int_{-\infty}^{\infty} dx\, \frac{e^{iax} - e^{ibx}}{x^2} = i\pi \mathrm{Res} \frac{e^{iaz} - e^{ibz}}{z^2}|_{z=0} = \pi(b-a)$$

$$\int_0^{\infty} dx\, \frac{\cos(ax) - \cos(bx)}{x^2} == \frac{\pi}{2}(b-a)$$

$$\int_0^{2\pi} \frac{d\theta}{4\cos\theta + 5}.$$

$$z = e^{i\theta}$$

$$\int_0^{2\pi} \frac{d\theta}{4\cos\theta + 5} = \int_{+\gamma} f(z)dz$$

$$f(z) = -\frac{i}{2z^2 + 5z + 2}.$$

$$\int_{+\gamma} f(z)dz = 2\pi i \mathrm{Res}(f, \frac{-1}{2}) = \frac{2}{3}\pi.$$

$$\int_0^{2\pi} \frac{d\theta}{(\cos\theta + 2)^2} = \int_{+\gamma} f(z)dz$$

$$f(z) = -\frac{4i\, z}{(z^2 + 4z + 1)^2}.$$

$$\int_{+\gamma} f(z)dz = 2\pi i \mathrm{Res}(f, -2 + \sqrt{3}) = 2\pi i \left(\frac{-2i}{3\sqrt{3}}\right) = \frac{4\pi}{3\sqrt{3}}.$$

$$\int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+5}\sin(2x)dx. = \text{Im}\int_{-\infty}^{\infty} \frac{x-2}{x^2-4x+5}e^{i2x}dx$$

Para hacer esta integral utilizamos el lema de Jordan. Escribimos $g(z) = \frac{z-2}{z^2-4z+5}$

E integramos $\int_{+\partial D_R} \mathrm{e}^{2iz} g(z)dz$ En el semiplano complejo superior

$$\int_{+\partial D_R} \mathrm{e}^{2iz} g(z)dz = 2\pi i \mathrm{Res}(\mathrm{e}^{2iz} g(z), 2+i) = 2\pi i\left(\frac{\mathrm{e}^{-2+4i}}{2}\right) = \pi \mathrm{e}^{-2+4i} i.$$

Separando la parte real e imaginaria y tomando solo esta última obtenemos

$$\int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+5}\sin(2x)dx = \frac{\pi}{\mathrm{e}^2}\cos(4).$$

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1} dx.$$

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1} dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{x^2+1} dx = \frac{\pi}{2} - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{x^2+1} dx.$$

Escribimos el coseno en términos de exponenciales

$$f(z) = \frac{e^{2\pi i z}}{z^2+1}$$

E integramos en el semiplano superior

$$\int_{+\partial D_R} f(z)dz = 2\pi i \operatorname{Res}(f, i).$$

$$\operatorname{Res}(f, i) = \lim_{z \to i} f(z)(z-i) = \frac{1}{2ie^{2\pi}}.$$

Separamos la parte real e imaginaria de la integral

$$\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{x^2+1} dx = \frac{\pi}{e^{2\pi}}.$$

Sumando todo

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1} dx = \frac{\pi}{2} - \frac{\pi}{2e^{2\pi}} = \frac{\pi}{2}\left(1 - \frac{1}{e^{2\pi}}\right).$$

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2}\,dx, \quad -1 < \alpha < 1$$

$$R(z) = \frac{1}{(1+z)^2}$$

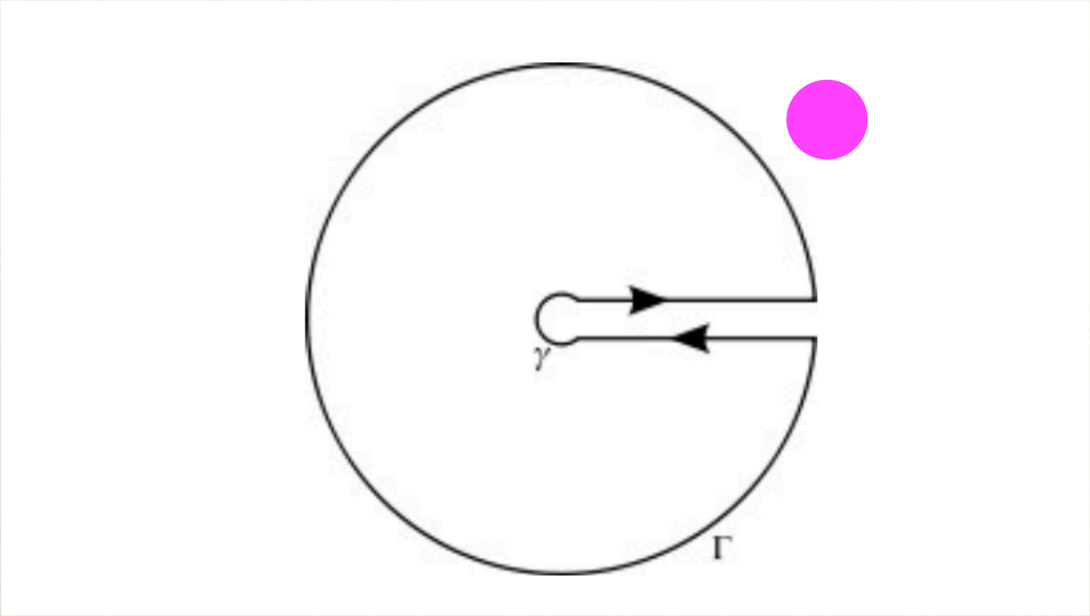
$$f(z) = R(z)z^\alpha = R(z)e^{\alpha \mathrm{Log}(z)}$$

Hay que definir bien el logaritmo para evitar ambigüedades

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg},\ z \quad 0 \leq \mathrm{Arg}\, z < 2\pi$$

$$f(x+i0) = f(x) = \frac{x^\alpha}{(1+x)^2} > 0 \qquad (x>0);$$
$$f(x-i0) = f(x)\mathrm{e}^{i2\pi\alpha} = \frac{x^\alpha \mathrm{e}^{i2\pi\alpha}}{(1+x)^2} > 0 \qquad (x>0).$$

Para aplicar el teorema de los residuos ahora necesitamos un circuito particular



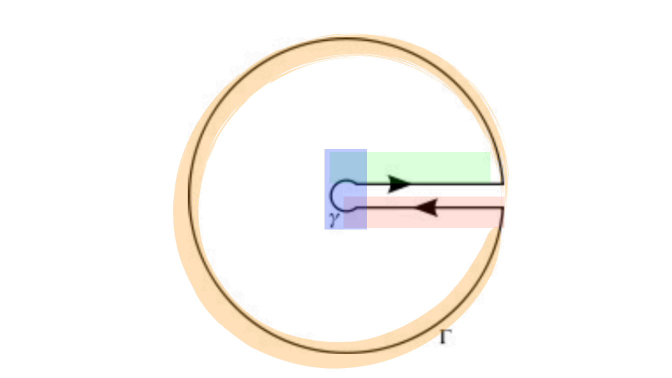
$$\begin{aligned} \rho &= \text{radio circunferencia pequeña} \\ R &= \text{radio circunferencia grande} \\ I &= \text{integral sobre todo el circuito} \\ I_\gamma &= \text{integral sobre circunferencia pequeña} \\ I_\Gamma &= \text{integral sobre semi-circunferencia grande} \end{aligned}$$

$$I = \text{integral sobre todo el circuito} = 2\pi i \sum \mathrm{Res}(R(z) z^{\alpha})$$

$$\mathrm{Res}(R(z)z^{\alpha}, -1) = \lim_{z \to -1} \frac{d}{dz} \frac{z^{\alpha}}{(1+z)^2}(z+1)^2 = \lim_{z\to -1} \frac{d}{dz} z^{\alpha} = \alpha\, e^{(\alpha-1)\, i\pi} = -\alpha\, e^{\alpha\, i\pi}.$$

$$I = \int_{\rho}^{R} f(x+i0)dx + \int_{+\Gamma_R} \frac{z^{\alpha}}{(1+z)^2}\, dz + \int_{R}^{\rho} f(x-i0)dx + \int_{+\gamma_\rho} \frac{z^{\alpha}}{(1+z)^2}\, dz$$

$$= (1 - e^{i2\pi\alpha}) \int_{\rho}^{R} \frac{x^{\alpha}}{(1+x)^2}\, dx + \int_{+\Gamma_R} \frac{z^{\alpha}}{(1+z)^2}\, dz + \int_{+\gamma_\rho} \frac{z^{\alpha}}{(1+z)^2}\, dz.$$



$$\left|\int_{+\gamma_\rho} \frac{z^\alpha}{(1+z)^2}\,dz\right| \le \int_{\gamma_\rho} \frac{|z|^\alpha}{(1-|z|)^2}\,ds_z \le 2\pi \frac{\rho^{\alpha+1}}{(1-\rho)^2} \xrightarrow{\rho\to 0} 0;$$

$$\left|\int_{+\Gamma_R} \frac{z^\alpha}{(1+z)^2}\,dz\right| \le \int_{\gamma_R} \frac{|z|^\alpha}{(|z|-1)^2}\,ds_z \le 2\pi \frac{R^{\alpha+1}}{(R-1)^2} \xrightarrow{R\to \infty} 0.$$

$$(1-\mathrm{e}^{i2\pi\alpha}) \int_0^\infty \frac{x^\alpha}{(1+x)^2}\,dx = 2\,\pi\, i(-\alpha\,\mathrm{e}^{\alpha\, i\,\pi})$$

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2}\,dx = \frac{\alpha\,\pi}{\sin(\alpha\pi)}.$$

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx, \quad 0<\alpha<4$$

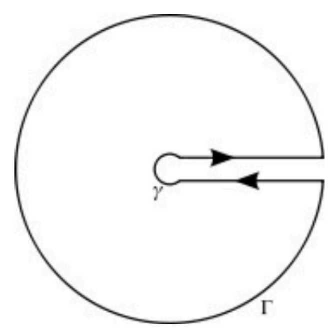
$$R(z) = \frac{1}{(1+z^2)^2}$$

$$f(z) = R(z)z^\alpha = R(z)e^{\alpha \mathrm{Log}(z)}$$

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg},\ z \quad 0 \le \mathrm{Arg}\, z < 2\pi$$

$$f(x+i0) = f(x) = \frac{x^{\alpha-1}}{(1+x^2)^2} > 0 \qquad (x>0);$$

$$f(x-i0) = f(x)\mathrm{e}^{i2\pi(\alpha-1)} = \frac{x^{\alpha-1}\mathrm{e}^{i2\pi\alpha}}{(1+x^2)^2} > 0 \qquad (x>0).$$



$$I_{\rho,R} = \int_{+\Gamma_{\rho,R}} \frac{z^{\alpha-1}}{(1+z^2)^2}\,dz = 2\pi i \sum \mathrm{Res}(R(z)z^{\alpha-1})$$

$$\operatorname{Res}(R(z)z^{\alpha-1}, i) = \lim_{z\to i} \frac{d}{dz}\frac{z^{\alpha-1}}{(1+z^2)^2}(z-i)^2 = \lim_{z\to i} \frac{d}{dz}\frac{z^{\alpha-1}}{(z+i)^2} = \frac{\alpha-2}{4}e^{i\alpha\pi/2}$$

$$\operatorname{Res}(R(z)z^{\alpha-1}, -i) = \lim_{z\to -i} \frac{d}{dz}\frac{z^{\alpha-1}}{(1+z^2)^2}(z+i)^2 = \lim_{z\to -i} \frac{d}{dz}\frac{z^{\alpha-1}}{(z-i)^2} = \frac{\alpha-2}{4}e^{3\,i\alpha\pi/2}$$

$$\begin{aligned} I_{\rho,R} &= \int_\rho^R f(x+i0)dx + \int_{+\Gamma_R} \frac{z^{\alpha-1}}{(1+z^2)^2}\,dz + \int_R^\rho f(x-i0)dx + \int_{+\gamma_\rho} \frac{z^{\alpha-1}}{(1+z^2)^2}\,dz \\ &= (1-e^{i2\pi\alpha}) \int_\rho^R \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx + \int_{+\Gamma_R} \frac{z^{\alpha-1}}{(1+z^2)^2}\,dz + \int_{+\gamma_\rho} \frac{z^{\alpha-1}}{(1+z^2)^2}\,dz. \end{aligned}$$

$$\lim_{\rho\to 0}\int_{+\gamma_\rho}\frac{z^{\alpha-1}}{(1+z^2)^2}\,dz = 0; \qquad \lim_{R\to\infty}\int_{+\Gamma_R}\frac{z^{\alpha-1}}{(1+z^2)^2}\,dz = 0.$$

$$\left|\int_{+\Gamma_R}\frac{z^{\alpha-1}}{(1+z^2)^2}\,dz\right| \le \int_{\gamma_R}\frac{|z|^{\alpha-1}}{(|z|^2-1)^2}\,ds_z \le 2\pi\frac{R^\alpha}{(R^2-1)^2} \xrightarrow{R\to\infty} 0.$$

$$\left|\int_{+\gamma_\rho}\frac{z^{\alpha-1}}{(1+z^2)^2}\,dz\right| \le \int_{\gamma_\rho}\frac{|z|^{\alpha-1}}{(1-|z|^2)^2}\,ds_z \le 2\pi\frac{\rho^\alpha}{(1-\rho^2)^2} \xrightarrow{\rho\to 0} 0$$

$$(1-e^{i2\pi\alpha})\int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx = 2\,\pi\,i\frac{\alpha-2}{4}(e^{i\alpha\pi/2}+e^{3\,i\alpha\pi/2}) = \pi i(\alpha-2)e^{i\alpha\pi}\cos(\alpha\pi/2)$$

$$\int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx = \frac{\pi(2-\alpha)\cos(\alpha\pi/2)}{2\,\sin(\alpha\pi)}.$$

Truco para integrales con logaritmos

Observar que $$\ln x = \lim_{\alpha \to 0} \frac{d}{d\alpha} x^{\alpha}$$

Entonces por ejemplo $$\int_0^\infty \frac{\ln x}{(1+x)^2} = \lim_{\alpha \to 0} \frac{d}{d\alpha} \int_0^\infty \frac{x^\alpha}{(1+x)^2}$$
$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} \frac{\alpha\pi}{\sin(\alpha\pi)} = 0$$

Truco para integrales con logaritmos

El truco se puede extender

$$\ln^n x = \lim_{\alpha\to 0} \frac{d^n}{d^n\alpha} x^\alpha$$

Ejemplo

$$\int_0^\infty \frac{\ln^n x}{(1+x)^2} = \lim_{\alpha\to 0} \frac{d^n}{d^n\alpha} \int_0^\infty \frac{x^\alpha}{(1+x)^2}$$

$$= \lim_{\alpha\to 0} \frac{d^n}{d^n\alpha} \frac{\alpha\pi}{\sin(\alpha\pi)}$$